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Hamilton relativity group for noninertial states in quantum mechanics

Stephen G Low

Austin, TX, USA

E-mail: Stephen.Low@alumni.utexas.net

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Abstract
Physical states in quantum mechanics are rays in a Hilbert space. Projective representations of a relativity group transform between the quantum physical states that are in the admissible class. The physical observables of position, time, energy and momentum are the Hermitian representation of the generators of the algebra of the Weyl–Heisenberg group. We show that there is a consistency condition that requires the relativity group to be a subgroup of the group of automorphisms of the Weyl–Heisenberg algebra. This, together with the requirement of the invariance of classical time, results in the inhomogeneous Hamilton group. The Hamilton group is the relativity group for noninertial frames in classical Hamilton's mechanics. The projective representation of a group is equivalent to unitary representations of the central extension of the group. The central extension of the inhomogeneous Hamilton group and its corresponding Casimir invariants are computed. One of the Casimir invariants is a generalized spin that is invariant for noninertial states. It is the familiar inertial Galilean spin with additional terms that may be compared to noninertial experimental results.

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1. Introduction

A relativity group defines a universal transformation between physical states in an admissible class. For special relativity, the inhomogeneous Lorentz group transforms inertial states that have relative rotation, velocity and translations in position and time into one another. It specifies how the measuring rods, clocks, momentum meters and energy meters of these states are related. For small velocities, the Lorentz group is approximated by the Euclidean group of Galilean relativity. In the quantum formulation, the physical states are realized as rays in a Hilbert space and the group acts through a projective representation. This is equivalent to the unitary representation of the central extension, both algebraic and topological, of the relativity group. The Poincaré group, that is the central extension of the inhomogeneous Lorentz group, is simply its cover as it does not have an algebraic extension [1]. The Galilei group is the central extension of the inhomogeneous Euclidean group that has a central mass generator as an algebraic extension.

We review projective representations on physical states that are rays in a Hilbert space and then show that this property of quantum mechanics leads directly to a consistency condition that relativity groups must be subgroups of the group of automorphisms of the Weyl–Heisenberg algebra. The Hamilton group is the subgroup of this group of automorphisms that leaves time invariant. In a previous paper, it was shown that the Hamilton group is the relativity group for noninertial frames in classical Hamilton's mechanics [2]. The projective representations of the inhomogeneous Hamilton group must be determined for its action on a quantum noninertial state. We will show that the central extension of the Hamilton group admits three central generators; I for the position-momentum and time–energy quantum commutation relations, M as the mass generator and a new central element A with dimensions that are the reciprocal of tension. The Galilei group is the subgroup of this group for inertial states and the mass generator is the same. The Casimir invariants are calculated and this leads to a generalized definition of Galilei spin invariant for noninertial physical states.

2. Projective representations of groups in quantum mechanics

Physical states are represented in quantum mechanics by rays Ψ in a Hilbert space $\mathbf{H}$. Rays are equivalence classes of states $|\psi\rangle \in \mathbf{H}$ that differ only in phase [3, 4]. Two states $|\psi\rangle, |\tilde{\psi}\rangle$ are in the same equivalence class Ψ if

$$|\tilde{\psi}\rangle = \mathrm{e}^{\mathrm{i}\omega}|\psi\rangle, \qquad \omega \in \mathbb{R}\,. \tag{1}$$

A symmetry of the physical system, described by a Lie group $\mathcal{G}$ with elements g, acts on the rays through a projective representation π to transform one physical state into another

$$\tilde{\Psi} = \pi(g)\Psi. \tag{2}$$

The projective representation has the property $\pi(g)\pi(\tilde{g}) = \mathrm{e}^{\mathrm{i}\omega(g,\tilde{g})}\pi(g\tilde{g}), \omega(g,\tilde{g}) \in \mathbb{R}$. As the rays are equivalence classes, $\Psi \simeq \mathrm{e}^{\mathrm{i}\omega(g,\tilde{g})}\Psi$ for any $\omega(g,\tilde{g}) \in \mathbb{R}$ and therefore

$$\pi(g)\pi(\tilde{g})\Psi = \pi(g\tilde{g})\Psi\,. \tag{3}$$

A class of observables is defined by lifting the projective representation to act on elements X of the Lie algebra $\mathbf{a}(\mathcal{G})$ so that $\pi'(X) = (T_e\pi)(X)$ is an operator on $\mathbf{H}$. An observable $\hat{\tilde{X}}$ acting on the state $\tilde{\Psi}$ is related to the observable $\hat{X}$ acting on the state Ψ by the projective representation of g

$$\pi(g)\hat{X}\Psi = \pi(g)\hat{X}\pi(g)^{-1}\pi(g)\Psi = \pi(g)\hat{X}\pi(g)^{-1}\tilde{\Psi} = \pi'(\tilde{X})\tilde{\Psi} = \hat{\tilde{X}}\tilde{\Psi}. \tag{4}$$

Therefore

$$\pi'(\tilde{X}) = \pi(g)\pi'(X)\pi(g)^{-1} = \pi'(gXg^{-1}) \tag{5}$$

for which a sufficient condition is[1]

$$\tilde{X} = gXg^{-1}. \tag{6}$$

Finally, we note that if an operator X is an invariant such that $\hat{\tilde{X}} = \hat{X}$, then

$$X = gXg^{-1}. \tag{7}$$

[1] If the representation is an isomorphism, then it is also necessary. Otherwise, equivalence is defined up to the equivalence class defined by taking the quotient of the group with the kernel of the representation.

A theorem in representation theory states that the projective representations π of a Lie group $\mathcal{G}$ are equivalent to the unitary or anti-unitary[2] representation ϱ of the central extension $\check{\mathcal{G}}$ [3] (see section 2.7 and appendix B of [1]). Therefore, the physical system may be studied by characterizing the unitary irreducible representations ϱ of the group $\check{\mathcal{G}}$ acting on a Hilbert space $\mathbf{H}^{\varrho}$. The Hilbert space is labeled by the unitary irreducible representation because it is not given *a priori*, but rather is determined by the unitary representation.

The unitary representations act on the states $|\psi\rangle$ in the Hilbert space $\mathbf{H}^{\varrho}$

$$|\tilde{\psi}\rangle = \varrho(g)|\psi\rangle, \qquad g \in \check{\mathcal{G}}. \tag{8}$$

The unitary representation ϱ may be lifted to the tangent space to define the Hermitian representation $\hat{X}$ of the element of the algebra X

$$\hat{X} = \varrho'(X) = T_e\varrho'(X). \tag{9}$$

Physical observables are characterized by the eigenvalues of the Hermitian representation of the generators

$$\hat{X}|\psi\rangle = x|\psi\rangle, \qquad x \in \mathbb{R}\,, \tag{10}$$

and these generators transform as

$$\varrho(g)\hat{X}|\psi\rangle = \varrho(g)\hat{X}\varrho(g)^{-1}|\tilde{\psi}\rangle = \hat{\tilde{X}}|\tilde{\psi}\rangle. \tag{11}$$

3. Quantum mechanics consistency condition with a relativity group

Measurements of the basic physical observables such as position, time, energy and momentum depend on the relative physical state in which they are measured. For certain classes of physical states, there is a relativity principle that defines a universal group relating the states. Examples were given in the introduction: the inhomogeneous Lorentz group, and its central extension the Poincaré group for the class of inertial quantum states in special relativistic quantum mechanics [1] and the inhomogeneous Euclidean group and its central extension the Galilei group for the class of inertial states in 'nonrelativistic' quantum mechanics[4].

In quantum mechanics, the observables of position, time, energy and momentum are the Hermitian representations of the algebra corresponding to the unitary representation of the Weyl–Heisenberg group $\mathcal{H}(n+1)$ [5].[5] It is the real matrix Lie group

$$\mathcal{H}(n+1) \simeq \mathcal{T}(n) \otimes_s \mathcal{T}(n+1), \tag{12}$$

where $\mathcal{T}(n) \simeq (\mathbb{R}^n, +)$. That is, $\mathbb{R}^n$ considered to be a Lie group under addition. The algebra has a basis $\{Z_\alpha\} = \{P_i, Q_i, E, T\} \in a(\mathcal{H}(n+1)), i, j, \ldots = 1, \ldots, n,$ and $\alpha, \beta, \ldots = 1, \ldots, 2n+2$. The algebra has the familiar commutation relations

$$[P_i, Q_j] = \delta_{i,j} I, \qquad [E, T] = -I. \tag{13}$$

The action of the group on the quantum states is given by the projective representation π that is the unitary representation ϱ of the central extension $\check{\mathcal{G}}$. The Weyl–Heisenberg is

[2] In what follows we will use only unitary with anti-unitary implicitly included.

[3] If $\mathcal{G} \simeq \check{\mathcal{G}}$, the group does not have any intrinsic projective representations and the projective representations are just the unitary representations.

[4] 'Nonrelativistic' means the approximation for velocities small relative to c. The 'nonrelativistic' has its own relativity group that is the Galilei group in the inertial case.

[5] This is also often called the Heisenberg group. It was Weyl who is generally credited with recognizing the commutation relations was a Lie algebra and determining the group.

a special case of the algebraic extension of the translation group $\mathcal{T}(2n+2)$. If X_α are the generators of the Abelian algebra of the translation group, then the central extension is

$$[X_\alpha, X_\beta] = M_{\alpha,\beta}, \qquad [X_\gamma, M_{\alpha,\beta}] = 0, \qquad M_{\alpha,\beta} = -M_{\beta,\alpha}\,. \tag{14}$$

The Weyl–Heisenberg group is the special case $M_{\alpha,\beta} = \zeta_{\alpha,\beta} I$, where $\zeta_{\alpha,\beta}$ is the skew symmetric symplectic metric (B.12). Now as discussed in appendix A, the central extension of a group may be constrained if it is the subgroup of a larger group. We know that position, time, energy and momentum are correctly realized in the quantum mechanics by the unitary representation of the Weyl–Heisenberg group and the associated Hermitian representation of its algebra and so must be constrained in this manner. Before turning to what this larger group is, we briefly review this familiar Hermitian representation.

The Mackey theorems for semidirect products (or the Stone von Neumann[6] theorem) may be used to compute the unitary irreducible representations. The results are well known. The representations are labeled by the eigenvalue of the single Casimir I, $\varrho'(I)|\psi\rangle = c|\psi\rangle$. The physical cases is $c = \hbar$ for which the Hilbert space is $\mathbf{H} \simeq L^2(\mathbb{R}^{n+1}, \mathbb{C})$. If we choose a basis $|q, t\rangle$ that diagonalizes $\hat{Q}_i$ and $\hat{T}$,

$$\begin{aligned}
&\langle q, t|\hat{Q}_i|\psi\rangle = q^i \psi(q, t), && \langle q, t|\hat{T}|\psi\rangle = -t\psi(q, t),\\
&\langle q, t|\hat{P}_i|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial q^i}\psi(q, t), && \langle q, t|\hat{E}|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}\psi(q, t),\\
&\langle q, t|\hat{I}|\psi\rangle = \hbar\psi(q, t).
\end{aligned} \tag{15}$$

The representations satisfy the quantum commutation relations[7]

$$[\hat{P}_i, \hat{Q}_j] = \mathrm{i}\hbar\delta_{i,j}, \qquad [\hat{E}, \hat{T}] = -\mathrm{i}\hbar. \tag{16}$$

Of course, one could equally well choose the momentum representation with basis $|p, t\rangle$ that diagonalizes $\hat{P}_i$ and $\hat{T}$ [6], or for that matter, a basis $|p, e\rangle$ that diagonalizes $\hat{P}_i, \hat{E}$ or $|q, e\rangle$ that diagonalizes $\hat{Q}_i, \hat{E}$.

We now return to the question of the larger group constraining the algebraic central extension. A basic consistency condition exists between the physical observables belonging to the algebra of the Weyl–Heisenberg group and the relativity group that acts on them. We consider the class of linear relativity groups that transform one state into another so that in both states, position, time, energy and momentum are represented by the Hermitian representation of the Weyl–Heisenberg algebra. Each observers' specific measurements of position, time, energy and momentum may differ, due to contractions and dilations of the relativity transformation. However, in any physical state that they are measured, they define a Weyl–Heisenberg algebra. Then, from (11), for $g \in \check{\mathcal{G}}$, we have

$$\varrho(g)\hat{Z}_\alpha|\psi\rangle = \varrho(g)\hat{Z}_\alpha\varrho(g)^{-1}\varrho(g)|\psi\rangle = \hat{Z}'_\alpha|\psi'\rangle. \tag{17}$$

Therefore, as in (4)–(6), it follows that for a faithful representation

$$\hat{Z}'_\alpha = \varrho'(Z'_\alpha) = \varrho(g)\hat{Z}_\alpha\varrho(g)^{-1} = \varrho'(gZg^{-1}) \tag{18}$$

and

$$Z'_\alpha = gZg^{-1}, \tag{19}$$

where we require that both $Z'_\alpha, Z_\alpha \in \mathbf{a}(\mathcal{H}(n+1))$. This means that g is an element of the automorphism group of the Heisenberg algebra, $\mathcal{A}ut_{\mathcal{H}}$ and therefore the relativity group $\mathcal{G}$

[6] The Stone von Neumann theorem is specific to the Weyl–Heisenberg group whereas the Mackey theorems apply to a general class of semidirect product groups.

[7] The i is inserted in the exponential relating group and algebra so that the algebra is represented by Hermitian rather than anti-Hermitian operators and this results in its appearance in the commutation relations (see [15]).

must be a subgroup, $\mathcal{G} \subset \mathcal{A}ut_{\mathcal{H}}$[8]. The automorphism group is given in [7] and in appendix B (B.8)–(B.11),

$$\mathcal{A}ut_{\mathcal{H}} \simeq \mathcal{A} \otimes_s \mathcal{Z}_2 \otimes_s \mathcal{HS}p(2n+2), \tag{20}$$

where

$$\mathcal{HS}p(2n+2) \simeq \mathcal{S}p(2n+2) \otimes_s \mathcal{H}(n+1). \tag{21}$$

Therefore we have the result that, to be consistent with quantum mechanics in which position, time, energy, and momentum are the Hermitian representation of the Weyl–Heisenberg group, the relativity group must be a subgroup of the automorphism group defined in (20)–(21).

This consistency may also be argued in the other direction. Suppose the position, time, energy and momentum degrees of freedom are described by the generators $\{Z_\alpha\} = \{P_i, Q_i, E, T\}$ that span the algebra of $\mathcal{T}(2n+2)$. This is the expected classical description where the extended phase space is $\mathbb{R}^{2n+2}$. The generators satisfy the commutation relations $[Z_\alpha, Z_\beta] = 0$. The quantum operators are the projective representations of the generators, $\pi'(Z_\alpha)$, with $Z_\alpha \in \mathbf{a}(\mathcal{T}(2n+2))$. The projective representation is equivalent to the unitary representation of the central extension of the group, $\varrho'(Z_\alpha)$ with $X_\alpha \in \mathbf{a}(\check{\mathcal{T}}(2n+2))$. Considered as a group by itself, the algebraic extension of the translation group is given in (14).

Suppose that a relativity group $\mathcal{G}$ relates the position, time, energy and momentum Hermitian operators in different physical states. The translation generators may be considered to be a subgroup of the inhomogeneous group $\mathcal{IG} = \mathcal{G} \otimes_s \mathcal{T}(2n+2)$. The group acting on the quantum states is the Hermitian representation of the algebra of the central extension $\mathcal{I}\check{\mathcal{G}}$. If we assume $\mathcal{G} \simeq \mathcal{S}p(2n+2)$ then

$$\mathcal{IG} \simeq \mathcal{IS}p(2n+2) \simeq \mathcal{S}p(2n+2) \otimes_s \mathcal{T}(2n+2). \tag{22}$$

The central extension of the inhomogeneous symplectic group is shown in appendix B (A.4)–(A.7) to be $\mathcal{I}\check{\mathcal{S}}p(2n+2) = \mathcal{HS}p(2n+2)$ defined in (21). The central elements for a central extension of the translation group (14) have been constrained by its embedding in the inhomogeneous symplectic group to be $M_{\alpha,\beta} = \zeta_{\alpha,\beta} I$.

Therefore, in this case, the relativity group restricts the admissible central extension of the translation group so that it is precisely the Weyl–Heisenberg group. This turns out to be also true for certain subgroups of the symplectic group $\mathcal{S}p(2n+2)$ that we shall turn to next. The system is quantized simply through the projective representation that is required because physical states are represented by rays in the Hilbert space.

To summarize, consistency of the relativity group and quantum mechanics requires that the relativity group is a subgroup of the group of automorphisms of the Weyl–Heisenberg algebra of the basic physical observables of position, time, energy and momentum.

On the other hand, we can start by assuming a relativity group that is the symplectic group, or certain subgroups, and that physical states are represented by rays in the Hilbert space with the corresponding projective representations of the group. Then it directly follows that the physical quantities of time, position, energy and momentum, represented classically by translation generators, are the Hermitian representations of the Weyl–Heisenberg algebra in the quantum formulation. The *quantization* is directly a result of the states being *rays* in a Hilbert space.

[8] If the observables are the Hermitian representation of the element of the algebra of another group, then the same arguments hold and the relativity group must be a subgroup of the automorphism group of that algebra.

4. Hamilton relativity group: invariance of time

Nonrelativistic quantum mechanics has the fundamental assumption that time is an invariant for inertial and noninertial physical states. All observers' clocks tick at the same rate. Time, position, energy and momentum are represented by the generators $Z_\alpha \in \mathbf{a}(\mathcal{H}(n+1))$, where $\{Z_\alpha\} = \{P_i, Q_i, E, T\}$. The requirement that a relativity group $\mathcal{G}$ leave T invariant is, for all $g \in \mathcal{G}$,

$$gTg^{-1} = T. \tag{23}$$

This condition, together with the requirements that the group is a subgroup of $\mathcal{A}ut_{\mathcal{H}}$, requires that the g be elements of the group

$$\mathcal{HS}p(n) \simeq \mathcal{Z}_2 \otimes_s \mathcal{S}p(2n) \otimes_s \mathcal{H}(n). \tag{24}$$

This is established in appendix B, (B.12), (B.13) with the matrix realization of $\mathcal{HS}p(n)$ given in (B.1).

This identity is also established in [2] and in that reference it is shown that the diffeomorphisms of the extended phase space $\mathbb{P} \simeq \mathbb{R}^{2n+2}$ into itself whose Jacobians are elements of the group $\mathcal{HS}p(n)$ are Hamilton's equations. The Weyl–Heisenberg subgroup of (24) is parameterized by the relative rate of change with time of position v^i, momentum f^i and energy r. That is velocity, force and power. Power is the central element. The generators of velocity are G_i, force F_i and power R.[9] A general element of this $\mathcal{H}(n)$ algebra is

$$v^i G_i + f^i F_i + rR. \tag{25}$$

The full classical relativity group including the time, position, energy and momentum generators is the group

$$\mathcal{IHS}p(n) \simeq \mathcal{HS}p(n) \otimes_s \mathcal{T}(2n+2). \tag{26}$$

For the action on the quantum physical states, we must determine the unitary representation of the central extension. The most general relativity group that has time as an invariant acting on a quantum theory is therefore given by the unitary representations of $\mathcal{I}\check{\mathcal{H}}\mathcal{S}p(n)$. The method for computing central extensions is reviewed in appendix A. It can be shown that the algebraic central extension requires the addition of the central element I that turns the translation group $\mathcal{T}(2n+2)$ in (24) into $\mathcal{H}(n+1)$.

A further simplification is possible that makes the physical meaning clearer by requiring invariance of the length $\delta^{i,j} Q_i Q_j$ in the inertial rest frame. This eliminates substantial mathematical complexity associated with non-orthonormal frames and enables the physical meaning to be more transparent. The inertial rest frame action of the group $\mathcal{HS}p(n)$ in (24) has the parameters $v^i = f^i = r = 0$ (see (B.1)) and so we need only consider the symplectic subgroup. Thus for $h \in \mathcal{S}p(2n)$

$$\delta^{i,j} Q_i Q_j = h \delta^{i,j} Q_i Q_j h^{-1}. \tag{27}$$

The required subgroup is $\mathcal{O}(n) \subset \mathcal{S}p(2n)$ (see [2] and appendix B) and it maps an orthonormal basis $\{Q_i\}$ into an orthonormal basis $\{\tilde{Q}_i\}$. This defines the Hamilton group

$$\mathcal{H}a(n) = \mathcal{Z}_2 \otimes_s \mathcal{O}(n) \otimes_s \mathcal{H}(n) \subset \mathcal{S}p(2n) \otimes_s \mathcal{H}(n), \tag{28}$$

that is the relativity group for transformations between frames in Hamilton's mechanics where the inertial rest position frames are orthonormal.

Again, the quantum theory requires us to consider the central extension of

$$\mathcal{IH}a(n) \simeq \mathcal{H}a(n) \otimes_s \mathcal{T}(2n+2). \tag{29}$$

[9] Note that the units of these are 1/velocity, 1/force and 1/power respectively.

As we will show in the following section, this central extension is of the form[10]

$$\begin{aligned}\mathcal{QH}a(n) = \mathcal{I}\check{\mathcal{H}}a(n) &\simeq \overline{\mathcal{H}a}(n) \otimes_s (\mathcal{T}(2) \otimes \mathcal{H}(n+1)) \\ &\simeq (\mathcal{Z}_2 \otimes \mathcal{Z}_2) \otimes_s \overline{\mathcal{SO}}(n) \otimes_s \mathcal{H}(n) \otimes_s (\mathcal{T}(2) \otimes \mathcal{H}(n+1)).\end{aligned} \tag{30}$$

Again, the central extension of the translation group in (26) is restricted by the inhomogeneous Hamilton relativity group precisely so that it defines the Weyl–Heisenberg subgroup required for the quantum realization of position, time, energy and momentum as the Hermitian realization of Heisenberg generators. We will show in the following section that it has three algebraic central generators. Like the Galilei group, it has a nontrivial algebraic central extension that is the mass generator M and furthermore has a second central element A that has the physical dimensions that are the reciprocal of tension (length/force) in addition to the central element I that appears in the $\mathcal{H}(n+1)$ subgroup . M and A are generators of the algebra of the $\mathcal{T}(2)$ translation subgroup that appears in (30).

5. Central extension of the inhomogeneous Hamilton algebra

The homogeneous and inhomogeneous Hamilton and Euclidean groups as well as the Weyl–Heisenberg group are real matrix Lie groups. The matrix realization of these groups is given in appendix B. The Lie algebras can therefore be directly computed from these matrix realizations. The resulting nonzero commutation relations of the algebra of $\mathcal{H}a(n)$ are

$$\begin{aligned}&[J_{i,j}, J_{k,l}] = J_{j,k}\delta_{i,l} + J_{i,l}\delta_{j,k} - J_{i,k}\delta_{j,l} - J_{j,l}\delta_{i,k},\\ &[J_{i,j}, G_k] = G_j\delta_{i,k} - G_i\delta_{j,k},\\ &[J_{i,j}, F_k] = F_i\delta_{j,k} - F_j\delta_{i,k},\\ &[G_i, F_k] = R\delta_{i,k}.\end{aligned} \tag{31}$$

The inhomogeneous Hamilton group $\mathcal{IH}a(n)$ requires the additional nonzero commutation relations:

$$\begin{aligned}&[J_{i,j}, Q_k] = -Q_j\delta_{i,k} + Q_i\delta_{j,k}, && [J_{i,j}, P_k] = -P_j\delta_{i,k} + P_i\delta_{j,k},\\ &[G_i, Q_k] = \delta_{i,k}T, && [F_i, P_k] = \delta_{i,k}T,\\ &[E, G_i] = -P_i, && [E, F_i] = Q_i,\\ &[E, R] = 2T. &&\end{aligned} \tag{32}$$

That T is a central element as expected is clear from the structure of the commutation relations. All physical states related by group transformations generated by this algebra leave T invariant. All these states have the same definition of time. A general element of the algebra is

$$Z = \alpha^{i,j} J_{i,j} + v^i G_i + f^i F_i + rR + q^i P_i + tE + p^i Q_i + eT. \tag{33}$$

The $\alpha^{i,j}$ are the $\frac{n(n-1)}{2}$ rotation angles, v^i velocity, f^i force, r power, q^i position, t time, p^i momentum and e energy. Correspondingly the generators have the dimensions such that Z is dimensionless.

5.1. Galilei group

Before continuing with the Hamilton group, we briefly review the inertial special case of the Hamilton group that is the familiar Euclidean group of Galilean relativity [2]. The Galilei

[10] $\mathcal{Z}_2 \otimes \mathcal{Z}_2$ is the four element parity, time reversal discrete group.

group is the central extension of the inhomogeneous Euclidean group $\mathcal{IE}(n)$

$$\mathcal{IE}(n) \simeq \mathcal{E}(n) \otimes_s \mathcal{T}(n+1) = \mathcal{SO}(n) \otimes_s \mathcal{T}(n) \otimes_s \mathcal{T}(n+1). \tag{34}$$

The algebra of $\mathcal{IE}(n) \subset \mathcal{IH}a(n)$ is spanned by the generators $\{J_{i,j}, G_i, P_i, E\}$ that are a subset of the full set of generators in (31), (32),

$$\begin{aligned} &[J_{i,j}, J_{k,l}] = J_{j,k}\delta_{i,l} + J_{i,l}\delta_{j,k} - J_{i,k}\delta_{j,l} - J_{j,l}\delta_{i,k}, \\ &[J_{i,j}, G_k] = G_j\delta_{i,k} - G_i\delta_{j,k}, \\ &[J_{i,j}, P_k] = -P_j\delta_{i,k} + P_i\delta_{j,k}, \\ &[E, G_i] = -P_i. \end{aligned} \tag{35}$$

The central extension $\mathcal{G}a(n) \simeq \mathcal{I}\check{\mathcal{E}}(n)$ may be directly computed using the method in appendix A. It is well known the algebraic central extension is the single generator M with the associated additional nonzero commutation relation

$$[G_i, P_k] = \delta_{i,k} M. \tag{36}$$

Note that M has the dimensions of mass and has a consistent physical interpretation as mass. The Galilei group may be written as

$$\mathcal{G}a(n) = (\mathcal{T}(1) \otimes \overline{\mathcal{SO}}(n)) \otimes_s \mathcal{H}(n), \tag{37}$$

where E is the generator of the algebra of the $\mathcal{T}(1)$ group, $J_{i,j}$ are the generators of the algebra of $\overline{\mathcal{SO}}(n)$ and $\{G_i, P_i, M\}$ generate the Weyl–Heisenberg group $\mathcal{H}(n)$ with M the central element. Of course, in the physical case $n = 3$, $\overline{\mathcal{SO}}(3) = \mathcal{SU}(2)$.

5.2. *Central extension of the inhomogeneous Hamilton group*

Returning to the central extension $\mathcal{I}\check{\mathcal{H}}a(n)$ of the Hamilton group, direct symbolic Lie algebra computation using *shape Mathematica* with the method described in appendix B results in the addition of the central elements

$$[P_i, Q_k] = \delta_{i,k} I, \qquad [E, T] = -I, \qquad [G_i, P_k] = \delta_{i,k} M, \qquad [F_i, Q_k] = \delta_{i,k} A, \tag{38}$$

to the Hamilton algebra defined in (31), (32). The three new central elements $\{M, A, I\}$, $(N_e = 3)$, result in the following terms being added to a general element of the algebra given in (33)

$$Z = \alpha^{i,j} J_{i,j} + v^i G_i + f^i F_i + rR + q^i P_i + tE + p^i Q_i + eT + aA + mM + \iota I. \tag{39}$$

The central extension condition for M in (38) is identical to the mass central extension in the Galilei group (36) and is precisely the condition for $\mathcal{G}a(n)$ to be the inertial subgroup of $\mathcal{QH}a(n)$. The central extension I, with dimensions of action, is precisely the condition for the unitary representation to yield the usual Heisenberg commutation relations. The final extension A is new and has the dimensions that is the reciprocal of tension, length/force. Therefore the parameter a has dimensions of tension, m of reciprocal mass and ι of reciprocal action.

6. Casimir invariants

Casimir invariant operators $C_\alpha, \alpha = 1, \ldots, N_c$ are polynomials in the enveloping algebra that commute with all the generators of the algebra of the group in question $[C_\alpha, Z_A] = 0$. $Z_A \in \mathbf{a}(\mathcal{G}), A = 1, \ldots, N_g$ and N_g is the dimension of $\mathcal{G}$. $\alpha = 1, \ldots, N_c$ where N_c is the number of Casimir invariants. The eigenvalues ν_α of the Hermitian representations of the Casimir invariants $\hat{C}_\alpha|\psi\rangle = \nu_\alpha|\psi\rangle, \nu_\alpha \in \mathbb{R}$ are invariants for all physical states related by the relativity group $\mathcal{G}$. These invariants typically label irreducible unitary representations (but not always completely) and represent fundamental physical quantities. For example, in both the Galilei and the Poincaré relativity group, mass and spin are the eigenvalues of the representations of the corresponding Casimir invariants.

The number N_c of Casimir invariants may be computed directly from a theorem that states that it is $N_c = N_g - N_r$. N_r is the rank of the $N_g \times N_g$ matrix $z^A c^C_{A,B}$ that is the adjoint representation of the algebra $[Z_A, Z_B] = c^C_{A,B} Z_C$ [8]. A general element of the algebra is $Z = z^A Z_A$. The Casimirs can be found by constructing a general element $p^l(Z_A)$ of the enveloping algebra up to a give order of polynomial in the generators with general coefficients. Setting $[p^l(Z_A), Z_A] = 0$ creates a linear set of equations in the coefficients that can be solved to determine the Casimir invariant operators. A central element is a polynomial of order 1 and so the first N_e Casimir invariants are the central elements. This is a conceptually a simple calculation but is best carried out using a symbolic computation package written in *Mathematica* [9] as the number of bracket computations and the linear equations is large for the number of algebras in question.

The results for the dimensions are given in the following table. Of course, while N_g may be determined for general n and N_e is independent of n, the N_c must be computed from the rank of the symbolic matrix on a case by case basis. For the remainder of this section we will restrict our attention to $n \leqslant 3$

$$\begin{array}{llllll} & N_g & N_e & N_c(n=1) & N_c(n=2) & N_c(n=3) \\ \mathcal{G}a(n): & \frac{1}{2}(n^2+3n+4) & 1 & 2 & 3 & 3 \\ \mathcal{I}\check{\mathcal{H}}a(n): & \frac{1}{2}(n^2+7n+12) & 3 & 4 & 5 & 5 \end{array} \tag{40}$$

Before considering the group $\mathcal{QH}a(n) = \mathcal{I}\check{\mathcal{H}}a(n)$ we briefly review the well-known results from the Galilei group, $\mathcal{G}a(n) \simeq \mathcal{I}\check{\mathcal{E}}(n)$. For $n \leqslant 3$, there are three Casimir invariants, one of which is the central element M,

$$\begin{aligned} C_1 &= M, & C_2 &= 2ME - P_i P_i, \\ C_3 &= M^2 S_{i,j} S_{i,j}, & S_{i,j} &= J_{i,j} - \frac{1}{M}(G_j P_i - G_i P_j). \end{aligned} \tag{41}$$

Note that C_3 is identically zero for $n = 1$. From (40), it follows that these are the complete set of Casimir invariants for $\mathcal{G}a(n)$ for $n \leqslant 3$. For clearer physical insight, note that the Casimir invariants C_2 and C_3 can be written as the energy and spin of an inertial (free) particle

$$E - E^\circ = \frac{1}{2M} P_i P_i, \qquad S^2 = S_{i,j} S_{i,j} \qquad \text{where} \qquad E^\circ = \frac{1}{2M} C_2. \tag{42}$$

The Casimir invariants of $\mathcal{QH}a(n)$ may be directly computed using the same method. For $n \leqslant 3$, there are five Casimir invariants, three of which are the central elements I, M, A,

$$\begin{aligned} &C_1 = I, \qquad C_2 = M, \qquad C_3 = A, \\ &C_4 = TT - IR, \\ &C_5 = C^2 B_{i,j} B_{i,j}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} C &= C_2 C_3 + C_4 = -AM + T^2 - IR, \\ B_{i,j} &= J_{i,j} + \frac{1}{C} D_{i,j}. \end{aligned} \tag{44}$$

C is a Casimir invariant as any polynomial combinations of a Casimir is a Casimir and the $D_{i,j}$ are given by

$$D_{i,j} = A D^1_{i,j} + M D^2_{i,j} + R D^3_{i,j} + I D^4_{i,j} + T\left(D^5_{i,j} + D^6_{i,j}\right), \tag{45}$$

where

$$\begin{aligned} D^1_{i,j} &= G_j P_i - G_i P_j, & D^2_{i,j} &= F_j Q_i - F_i Q_j, \\ D^3_{i,j} &= P_i Q_j - P_j Q_i, & D^4_{i,j} &= F_i G_j - F_j G_i, \\ D^5_{i,j} &= F_i P_j - F_j P_i, & D^6_{i,j} &= G_i Q_j - G_j Q_i. \end{aligned} \tag{46}$$

$B_{i,j}$ vanishes for $n = 1$. It is a straightforward computation using the Lie algebra relations (31), (32), (38) to verify that these are invariant. From (40), it follows that these are the complete set of Casimir invariants for $\mathcal{QH}a(n)$ for $n \leqslant 3$. Note that $B_{i,j}$ may also be written as $B_{i,j} = S_{i,j} + \frac{1}{C}\tilde{D}_{i,j}$, where $S_{i,j}$ is the Galilean spin defined in (41) and as $\frac{1}{M} + \frac{A}{C} = \frac{C_4}{C}$

$$\tilde{D}_{i,j} = C_4 D^1_{i,j} + M D^2_{i,j} + R D^3_{i,j} + I D^4_{i,j} + T\left(D^5_{i,j} + D^6_{i,j}\right). \tag{47}$$

The eigenvalues of the Casimirs in the unitary representation usually label irreducible representations. In a representation where the eigenvalue of C goes to zero, the $D_{i,j}$ term will be negligible and the spin reduces to the usual Galilean spin $\lim_{C\to 0} B_{i,j} = S_{i,j}$. A sufficient condition for this is $C_3 = A \to 0$ and $C_4 \to 0$. As $A = 0$ means that the tension $1/A$ is infinite.

7. Discussion

Physical states in quantum mechanics are represented by rays in a Hilbert space. Quantum mechanics realizes position, time, momentum and energy as the Hermitian representation of the generators $\hat{Z}_\alpha$ of the algebra of the Weyl–Heisenberg group. These generators acting on a state, $\hat{Z}_\alpha|\psi\rangle$, are transformed by the unitary representations of a relativity group to define generators acting on the transformed state, $\tilde{\hat{Z}}_\alpha|\tilde{\psi}\rangle$. In order for the transformed generators to also be generators of the Weyl–Heisenberg group, the relativity group must be a subgroup of the group of automorphisms of the Weyl–Heisenberg algebra. If the relativity group does not have this property, then the position, time, momentum and energy degrees of freedom would not satisfy the Heisenberg commutation relations in the transformed state, as given in (16). This provides a basic consistency condition between the relativity group and the Weyl–Heisenberg group of quantum mechanics.

Folland has proven that the automorphism group of the Weyl–Heisenberg algebra (and group) is the group $\mathcal{HS}p(2n+2)$ together with a conformal scaling group $\mathcal{A}$ and a two element discrete group $\mathcal{Z}_2$ that reverses the sign of time and energy. The Weyl–Heisenberg subgroup of $\mathcal{HS}p(2n+2)$ are the inner automorphisms. The constraint on the continuous homogeneous relativity group is that it is a subgroup of the symplectic subgroup and conformal scaling group. This shows the very deep connection between the Weyl–Heisenberg group and the symplectic structure. The symplectic structure does not need to be postulated *shape a priori* but is simply required by this consistency with quantum mechanics.

Invariance of classical time results in the $\mathcal{HS}p(2n)$ homogeneous relativity group that is a subgroup of $\mathcal{S}p(2n+2)$. We have previously shown that the requirement that Jacobians of diffeomorphisms of classical extended phase space into itself be elements of this group are Hamilton's equations. While this is the most general group, a considerable amount of the mathematical generality of this group is to accommodate frames that are not orthonormal in the inertial rest frame. Requiring invariance of length in the inertial rest frame reduces this group to the physical Hamilton group $\mathcal{H}a(n)$ [2].

The central generators of mass, fundamental to classical mechanics, and the I required to give the Heisenberg commutation relations in the representation appear automatically as a result of the projective representations that are required because the physical quantum states are rays in a Hilbert space.

Projective representations of this group act on quantum states that are generally noninertial. Here we get our first surprise. In the quantum realization, time is not an invariant. Rather, taking the representation of (43) we obtain an expression of the form acting on a quantum state

$$T^2|\psi\rangle = (\tau^2 + \hbar R)|\psi\rangle, \tag{48}$$

where τ^2 is the eigenvalue of the Casimir C_4.

The next interesting fact is that there is no Casimir operator (such as C_2 for the Galilei group) that involves the energy generator. However, when one recalls from basic classical mechanics that noninertial frames do not conserve energy, this is not so surprising and to be expected. What is surprising is that there is a natural generalization of spin that is invariant in all noninertial physical states. All observers, inertial and noninertial, calculate the same values for the eigenvalues of the Casimir C_5 that may be expressed as Galilei spin with additional terms (44), (45). This provides a possibility of testing this theory by studying spin in noninertial states in 'nonrelativistic' quantum mechanics.

Finally, an even more surprising result is the appearance of a new central generator A. The parameter a for the term aA in the general term for the algebra (39) has the dimensions of tension. This generator shows up in the algebra in as fundamental a manner as the mass and I generator. Both of those are rather basic to physics and so a critical test of these ideas is a further physical understanding of A. Note A is reciprocally dual to M in the sense of Born [10]. This reciprocal symmetry underlies the material presented.

As noted, the eigenvalues of the Hermitian representations of the Casimir invariants for the unitary representations of the group typically label the irreducible representations. There will probably be a class of irreducible representations with states where $\hat{A}|\psi\rangle = 0$. In these states *tension* is infinite. Tension seems to play a basic role in string theory and so it is intriguing that a tension term shows up here.

The unitary representations of the group $\mathcal{QH}a(n)$, and many of the other groups in this paper, have a rich semidirect product structure. The methods of Mackey for determining the unitary representations of semidirect product groups can be used to determine the representations [11]. This will be undertaken in a subsequent paper to complete the quantum description of this relativity group.

The same methods may be generalized to relativity groups with other invariants. For example, instead of the invariant T, we could have the Minkowski invariant $T^2 - \frac{1}{c^2}Q^2$ appropriate for special relativity or the Born line element $T^2 - \frac{1}{c^2}Q^2 - \frac{1}{b^2}P^2 + \frac{1}{c^2b^2}E^2$ appropriate for reciprocal relativity [12].[11] In particular, one would conjecture a generalization of the standard spin in relativistic quantum mechanics that involves additional terms similar to what has been shown here in the classical context required for noninertial frames to provide a critical test.

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Appendix A. Central extensions

The central extension $\check{\mathcal{G}}$ of a group $\mathcal{G}$ is defined by the short exact sequence

$$1 \to \mathcal{A} \to \check{\mathcal{G}} \to \mathcal{G} \to 1,$$

[11] We note that $\mathcal{QH}a(n)$ is the $b, c \to \infty$ limit of the quaplectic group that appears in reciprocal relativity.

where $\mathcal{A}$ is an Abelian group in the center of $\check{\mathcal{G}}$ that is the extension. This induces a central extension for the algebra of $\check{\mathcal{G}}$ corresponding to the algebra of $\mathcal{A}$. The set of isomorphism classes of the central extension of $\mathcal{G}$ by $\mathcal{A}$ is in one-to-one correspondence with the second cohomology group $H^2(\mathcal{G}, \mathcal{A})$. The methods of algebraic topology that may be used to determine this cohomology group are described in [13].

Alternatively, a central extension $\check{\mathcal{G}}$ is the universal cover of the group whose algebra is the central extension of the algebra of $\mathcal{G}$. The central extension of the algebra is explicitly constructed as the most general central extension satisfying the Jacobi identities for the algebra as described in [1]. A nontrivial first homotopy group for the group $\mathcal{G}$ results in the topological extension that is the first homotopy group in the construction of the universal cover. We refer to the extension of the algebra as the algebraic extension and the cover as the topological extension. This indirectly results in the cohomology group $H^2(\mathcal{G}, \mathcal{A})$ due to the equivalence noted. This method of determining the algebraic extension is tractable by creating a set of general Lie algebra evaluation rules in *Mathematica* [9].

Consider a general Lie algebra with basis $Z_\alpha \in \mathbf{a}(\mathcal{G}), \alpha, \beta, \ldots = 1, \ldots, N_g$, with N_g the dimension of the group and algebra, satisfying commutation relations

$$[Z_\alpha, Z_\beta] = c^{\gamma}_{\alpha,\beta} Z_\gamma. \tag{A.1}$$

A general element of the algebra is $Z = z^\alpha Z_\alpha$ with $z^\alpha \in \mathbb{R}$. The central extension of the algebra is defined by the maximal addition of central elements $M_{\alpha,\beta}$ to the algebra that are consistent with the Jacobi identities. First, construct the maximal set of candidate extensions as

$$[Z_\alpha, Z_\beta] = c^{\gamma}_{\alpha,\beta} Z_\gamma + M_{\alpha,\beta}, \qquad [Z_\gamma, M_{\alpha,\beta}] = 0. \tag{A.2}$$

The problem of determining the central extension is to find the most general set of $M_{\alpha,\beta}$ for which the Jacobi identities for the set of generators are satisfied

$$[Z_\alpha, [Z_\beta, Z_\gamma]] + [Z_\beta, [Z_\gamma], Z_\alpha]] = [Z_\gamma, [Z_\alpha, Z_\beta]] = 0. \tag{A.3}$$

Clearly the combination $M_{\alpha,\beta} = c^{\gamma}_{\alpha,\beta} M_\gamma$ is always a solution constituting to translating each generator by a corresponding central element, $\check{Z}_\alpha = Z_\alpha + M_\alpha$. These are discarded as trivial. The remaining solutions define the central extension.

Consider a Lie group $\mathcal{G}$ with a candidate central extension of its algebra given by (A.2). The number of Jacobi identities can be dramatically reduced by examining the properties of the central extensions for subsets of the generators $\{Z_a\}$ that define subalgebras. A necessary condition that the full algebra admits a central extension is that the subalgebra also admit the extension. If the subalgebra does not have an extension, we can immediately set the corresponding set of candidate central elements to zero $\{M_{a,b} = 0\}$. The embedding in the larger group in these cases blocks the extension from being an extension of the full group.

It is well known that the following groups do not have an algebraic central extension: $\mathcal{O}(n), \mathcal{O}(1,n), \mathcal{S}p(2n), \mathcal{E}(n), \mathcal{E}(1,n)$. (See, for example, [1] for a detailed analysis of $\mathcal{E}(1,n)$.) $\mathcal{E}(n) = \mathcal{SO}(n) \otimes_s \mathcal{T}(n)$ provides an immediate example where the subgroup $\mathcal{T}(n)$ has a central extension that is blocked by the fact that it is a subgroup of the semidirect product with $\mathcal{SO}(n)$ as the homogeneous group.

The algebraic central extension of the inhomogeneous symplectic group may be calculated in the same manner. Suppose $\{W_{a,b}, Y_a\}$ are the generators of the algebra of the inhomogeneous symplectic group, $\mathcal{IS}p(2n+2) \simeq \mathcal{S}p(2n+2) \otimes_s \mathcal{T}(2n)$. The nonzero commutation relations are

$$[W_{\alpha,\beta}, W_{\kappa,\delta}] = \zeta_{\beta,\kappa} W_{\alpha,\delta} + \zeta_{\alpha,\kappa} W_{\beta,\delta} + \zeta_{\beta,\delta} W_{\alpha,\kappa} + \zeta_{\alpha,\delta} W_{\beta,\kappa}, \tag{A.4}$$

$$[W_{\alpha,\beta}, Y_\kappa] = \zeta_{\beta,\kappa} Y_\kappa + \zeta_{\alpha,\kappa} Y_\kappa, \tag{A.5}$$

where $\alpha, \beta, \ldots = 1, \ldots, 2n+2$. Immediately $M^{W,W}_{\alpha,\beta,\kappa,\delta} = 0$ as $\mathcal{S}p(2n+2)$ does not admit a central extension. It is then simply a matter of introducing the candidate extensions $\{M^{Y,Y}_{\alpha,\beta}, M^{W,Y}_{\alpha,\beta,\kappa}\} = \{M_{\alpha,\beta}, M_{\alpha,\beta,\kappa}\}$ and checking the Jacobi relations when they are added to the above generators and the implicit relation

$$[Y_\alpha, Y_\beta] = M_{\alpha,\beta}. \tag{A.6}$$

The essential Jacobi relations from (A.3) are

$$\begin{aligned}\{Y_\alpha, W_{\kappa,\delta}, W_{\gamma,\epsilon}\} &= \zeta_{\epsilon,\kappa} M_{\gamma,\delta,\alpha} + \zeta_{\alpha,\kappa} M_{\gamma,\epsilon,\delta} + \zeta_{\alpha,\delta} M_{\gamma,\epsilon,\kappa} \\ &\quad - \zeta_{\delta,\epsilon} M_{\gamma,\kappa,\alpha} + \zeta_{\gamma,\kappa} M_{\delta,\epsilon,\alpha} - \zeta_{\alpha,\epsilon} M_{\delta,\kappa,\gamma} - \zeta_{\alpha,\gamma} M_{\delta,\kappa,\epsilon} + \zeta_{\gamma,\delta} M_{\epsilon,\kappa,\alpha}, \\ \{Y_\alpha, Y_\kappa, W_{\gamma,\epsilon}\} &= -M_{\kappa,\epsilon}\zeta_{\alpha,\gamma} - M_{\kappa,\gamma}\zeta_{\alpha,\epsilon} - M_{\alpha,\epsilon}\zeta_{\gamma,\kappa} - M_{\alpha,\gamma}\zeta_{\epsilon,\kappa}.\end{aligned} \tag{A.7}$$

It can be verified that $M_{\gamma,\delta,\alpha}$ only has trivial solutions whereas $M_{\alpha,\beta} = \zeta_{\alpha,\beta} I$ where I is a central element is a nontrivial solution. Thus, the algebraic central extension is the group $\mathcal{HS}p(n) \simeq \mathcal{S}p(2n) \otimes_s \mathcal{H}(n)$ as claimed.

As another example, this method can be immediately used to reduce the number of central generators that need to be checked with the Jacobi conditions for the Galilei group. Here the candidate central elements (with superscripts labeling the commutators to which they apply) are $\{M^{J,J}_{i,j,k,l}, M^{J,G}_{i,j,k}, M^{J,P}_{i,j,k}, M^{J,E}_{i,j}, M^{G,E}_{i}, M^{P,E}_{i}, M^{G,P}_{i,j}\}$. As $\{J_{i,j}, G_i\}$ and $\{J_{i,j}, P_i\}$ are the generators of Euclidean subalgebras, we can immediately set $M^{J,J}_{i,j,k,l} = 0$, $M^{J,G}_{i,j,k} = 0$, $M^{J,P}_{i,j,k} = 0$ significantly reducing the number of Jacobi identities that need to be calculated.

The same is true of the inhomogeneous Hamilton group where the relations $\{M^{J,F}_{i,j,k}, M^{J,Q}_{i,j,k}\}$ may also be set to zero as they are the generators of Euclidean subalgebras. The remainder need to be checked directly through the Jacobi identities that is best undertaken using the symbolic computation capabilities of *Mathematica.*

Appendix B. Matrix realizations of the groups

The homogeneous and inhomogeneous Hamilton and Euclidean groups and the Weyl–Heisenberg groups may be realized as matrix groups that are subgroups of $\mathcal{GL}(2n+2, \mathbb{R})$. Elements of this group are nonsingular $(2n+2) \times (2n+2)$ real matrices.

The group element $\Gamma^\circ(\epsilon, A, w, \iota) \in \mathcal{HS}p(2n)$ and Hamilton group $\Phi^\circ(\epsilon, R, v, f, \iota) \in \mathcal{H}a(n)$ are matrix subgroups of $\mathcal{GL}(2n+2, \mathbb{R})$ [2, 14] with the form

$$\Gamma^\circ = \begin{pmatrix} A & 0 & w \\ -{}^t w \zeta^\circ A & \epsilon & r \\ 0 & 0 & \epsilon \end{pmatrix}, \qquad \Phi^\circ = \begin{pmatrix} R & 0 & 0 & f \\ 0 & R & 0 & v \\ {}^t v R & -{}^t f R & \epsilon & r \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \tag{B.1}$$

where $r \in \mathbb{R}$, $\epsilon = \pm 1$ and $w \in \mathbb{R}^{2n}$, $\epsilon = \pm 1$, $A \in \mathcal{S}p(2n)$ realized by $2n \times 2n$ matrices in Γ° and $f, v \in \mathbb{R}^n$, $R \in \mathcal{O}(n)$ realized by $n \times n$ matrices in Φ°. The symplectic metric is

$$\zeta^\circ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{B.2}$$

where I_n is the $n \times n$ unit matrix. The subgroup chain,

$$\mathcal{H}(n) \subset \mathcal{H}a(n) \subset \mathcal{HS}p(2n) \subset \mathcal{GL}(2n+2, \mathbb{R}), \tag{B.3}$$

leads to the identifications of $w = (f, v)$ and also note that $\mathcal{O}(n) \subset \mathcal{S}p(2n)$.

Elements of the Weyl–Heisenberg group are given by either $\Upsilon(w, \iota) = \Gamma^\circ(1, I_{2n}, w, \iota)$ or $\Upsilon(v, f, \iota) = \Phi^\circ(1, I_n, v, f, \iota)$. The group multiplication, inverses and automorphisms may be computed simply through matrix multiplication and inverse. The basis of the Lie algebra is given by differentiating the matrices by the parameters and evaluating at the identity. The Lie algebra structure relations are computed directly by matrix multiplication to establish the abstract relations in (31).

Likewise, for the inhomogeneous group, we have the inclusion chain

$$\mathcal{IE}(n) \subset \mathcal{IHa}(n) \subset \mathcal{IHS}p(2n+2) \subset \mathcal{IGL}(2n+2,\mathbb{R}) \subset \mathcal{GL}(2n+3,\mathbb{R}). \tag{B.4}$$

The corresponding matrix representations of the inhomogeneous groups $\Gamma(\epsilon, A, w, r, z, e, \iota) \in \mathcal{IHS}p(2n+2)$ and $\Phi(\epsilon, R, v, f, r, q, p, e, t) \in \mathcal{IHa}(n)$ are

$$\Gamma = \begin{pmatrix} A & 0 & w & z \\ -{}^t w\zeta^\circ A & \epsilon & r & e \\ 0 & 0 & \epsilon & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} R & 0 & 0 & f & p \\ 0 & R & 0 & v & q \\ {}^t vR & -{}^t fR & \epsilon & r & e \\ 0 & 0 & 0 & \epsilon & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.5}$$

In these expressions $z \in \mathbb{R}^{2n}$, $p, q \in \mathbb{R}^n$ and there is the identification $z = (p, q)$. Elements of the form $\Phi(\epsilon, R, v, 0, 0, p, 0, e, 0)$ define an inhomogeneous Euclidean subgroup $\mathcal{IE}(n)$, the central extension of which is the Galilei group.

Finally, $\mathcal{HS}p(n) \simeq \mathcal{S}p(2n+2) \otimes_s \mathcal{H}(n)$ is a matrix subgroup of $\mathcal{GL}(2n+4,\mathbb{R})$

$$\Phi(\epsilon, A, w, \iota, z, e, t) = \begin{pmatrix} A & 0 & w & 0 & z \\ -{}^t w\zeta^\circ A & \epsilon & \iota & 0 & e \\ 0 & 0 & \epsilon & 0 & t \\ -{}^t z\zeta^\circ A & -t & e & 1 & \iota \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.6}$$

The central extension of a matrix group is not necessarily a matrix group [15]. It is not been established whether the extensions $\check{\mathcal{H}}\mathcal{S}p(n)$ and $\mathcal{QHa}(n)$ are matrix groups.

B.1. Automorphisms of the Weyl–Heisenberg group

$\mathcal{A}ut_{\mathcal{H}}$ that is given by (20) is proven by Folland. (See page 20 of [7].) We provide here the matrix representation and group composition law and explicitly compute the automorphisms. Using the definition $\Upsilon(w, \iota) = \Gamma(I_{2n}, w, \iota)$ and (B.1) the Weyl–Heisenberg group $\mathcal{H}(n)$ composition law is the expected

$$\Upsilon(w', \iota') \cdot \Upsilon(w, \iota) = \Upsilon(w + w', \iota + \iota' + {}^t w'\zeta^\circ w), \qquad \Upsilon(w, \iota)^{-1} = \Upsilon(-w, -\iota). \tag{B.7}$$

Elements of the linear automorphism group $\mathcal{A}ut_{\mathcal{H}}$ may be represented by $(2n+2)\times(2n+2)$ matrices Ω that satisfy $\Omega\Upsilon(w', \iota')\Omega^{-1} = \Upsilon(w'', \iota'')$, where $\Upsilon(w, \iota)$ are realized by $(2n+2)\times(2n+2)$ matrices (B.1). Direct computation then shows that the most general element with this property is

$$\Omega(\epsilon, a, A, w, r) = \begin{pmatrix} aA & 0 & w \\ -{}^t w\zeta^\circ A & \epsilon a^2 & r \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{B.8}$$

where $A \in \mathcal{S}p(2n)$, $w \in \mathbb{R}^{2n}$, $a, r \in \mathbb{R}$, $\epsilon = \pm 1$ and ζ° is the symplectic metric defined in (B.2). The group multiplication and inverse are

$$\begin{aligned} \Omega(\epsilon'', a'', A'', w'', r'') &= \Omega(\epsilon, a, A, w, r)\Omega(\epsilon', a', A', w', r') \\ &= \Omega(\epsilon\epsilon', aa', AA', \epsilon' w + aAw', \epsilon' r + \epsilon a^2 r' -{}^t w\zeta^\circ Aw'), \\ \Omega(\epsilon, a, A, w, r)^{-1} &= \Omega(\epsilon, a^{-1}, a^{-1}A^{-1}, -\epsilon a^{-1}A^{-1}w, -a^{-2}r). \end{aligned} \tag{B.9}$$

It follows directly that the automorphisms are

$$\begin{aligned}\Upsilon(w'', r'') &= \Omega(\epsilon', a', A', w', r')\Upsilon(w, r)\Omega(\epsilon', a', A', w', r')^{-1} \\ &= \Upsilon(\epsilon' a' A' w, a'^2 r - \epsilon'^t w' \zeta A' w + a'^t (A' w)\zeta w'). \end{aligned} \tag{B.10}$$

The central extension of the automorphism group of the Weyl–Heisenberg group is

$$\check{\mathcal{A}ut}_{\mathcal{H}} \simeq \overline{\mathcal{A}ut}_{\mathcal{H}} = (\mathcal{A} \otimes \mathcal{Z}_2) \otimes_s \overline{\mathcal{HS}p}(2n+2), \tag{B.11}$$

where $\overline{\mathcal{HS}p}(2n+2) \simeq \overline{\mathcal{S}p}(2n+2) \otimes_s \mathcal{H}(n+1)$. Folland also shows that the automorphism group for the Weyl–Heisenberg algebra and group are the same [7]. Note that the central generator I of the algebra of $\mathcal{H}(n+1)$ is also a central element of $\check{\mathcal{A}ut}_{\mathcal{H}}$.

B.2. Invariance of time

The subgroup $\mathcal{G}$ of the homogeneous subgroup of $\mathcal{A}ut_{\mathcal{H}}$, $g \in \mathcal{G} \subset \mathcal{A} \otimes_s \mathcal{Z}_2 \otimes_s \mathcal{S}p(2n+2)$, such that $gTg^{-1} = T$ is established directly through a matrix realization of the symplectic group and Weyl–Heisenberg algebra. First, note that a matrix $S \in \mathcal{S}p(2n+2)$ leaves invariant the symplectic metric $S\zeta^t S = \zeta$. The matrix for ζ and the matrix representation for the Lie algebra generator T are realized as $(2n+2) \times (2n+2)$ matrices as follows in the basis ordering $\{Z_\alpha\} = \{P_i, Q_i, E, T\}$:

$$\zeta = \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.12}$$

ζ° is defined in (B.2).[12] Using $S^{-1} = -\zeta^t S\zeta$, a general matrix element S and S^{-1} of $\mathcal{S}p(2n+2)$ have the form [16]

$$S = \begin{pmatrix} B & b_1 & b_2 \\ c_1 & d_{1,1} & d_{1,2} \\ c_2 & d_{2,1} & d_{2,2} \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} B^{-1} & -\zeta^{\circ\, t} c_2 & \zeta^{\circ\, t} c_1 \\ {}^t b_2 \zeta^\circ & d_{2,2} & -d_{1,2} \\ -{}^t b_1 \zeta^\circ & -d_{2,1} & d_{1,1} \end{pmatrix}^{-1}. \tag{B.13}$$

B is a $2n \times 2n$ matrix, $b_\alpha, c_\alpha \in \mathbb{R}^{2n}$ and $d_{\alpha,\beta} \in \mathbb{R}, \alpha, \beta = 1, 2$. Imposing the condition $T = STS^{-1}$ through matrix multiplication results in $b_1 = 0, d_{2,1} = 0$ and $d_{1,1} = \epsilon, \epsilon = \pm 1$. Finally, imposing the condition $SS^{-1} = I_{2n+2}$ results in $c_2 = 0, d_{2,2} = \epsilon$ and $c_1 = -\epsilon^t b_2 \zeta^\circ A$. Therefore, elements of $\mathcal{S}p(2n+2)$ that leave T invariant have the form $\Gamma^\circ \in \mathcal{HS}p(n)$ are given above in (B.1). Similar arguments result in the invariance of ${}^t Q_i Q_i$ to reduce the form further to $\Phi^\circ \in \mathcal{H}a(n)$.

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[12] The relative sign of ζ° and the 2×2 symplectic submatrix is a convention. An automorphism takes one form into the other. Note that $-de \wedge dt + dp \wedge dq = dt \wedge de + dp \wedge dq$ by the properties of the wedge product.

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